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On diffeomorphisms over non-orientable surfaces embedded in the 4-sphere

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1. Introduction

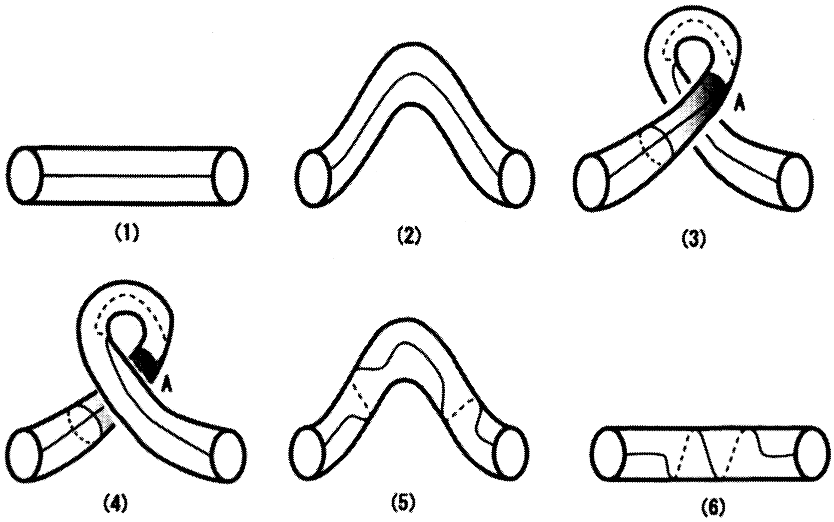

Figure 1

We put an annulus in $\mathbb{R}^4$, and deform this in $\mathbb{R}^4$ with fixing its boundary as shown in Figure 1. We can change crossing from (3) to (4) because this annulus is in $\mathbb{R}^4$. After this deformation, this annulus is twisted two times along the core. This means that this double twist can be extended to the ambient $\mathbb{R}^4$. In this note, we will discuss how many diffeomorphisms over the embedded surface are extendable to the ambient 4-space.

For some special embeddings of closed surfaces in 4-manifolds, we have answers to the above problem (for example, [9], [3], [4]). An embedding e of the orientable surface Σ_g into S^4 is called standard if there is an embedding of 3-dimensional handlebody into S^4 such that whose boundary is the image of e. In [9] and [3], we showed:

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Theorem 1.1 ([9] ($g = 1$), [3] ($g \geq 2$)). *Let Σ_g be standardly embedded in S^4. A diffeomorphism ϕ over the Σ_g is extendable to S^4 if and only if ϕ preserves the Rokhlin quadratic form of the Σ_g.*

In this note, we will introduce some approach to the same kind of problem for non-orientable surfaces embedded in S^4.

2. Setting

Let N_g be a connected non-orientable surface constructed from g projective planes by connected sum. We call N_g *the closed non-orientable surface of genus g*. For a smooth embedding e of N_g into S^4, Guillou and Marin ([2] see also [8]) defined a quadratic form $q_e : H_1(N_g; \mathbb{Z}_2) \to \mathbb{Z}_4$ as follows: Let C be an immersed circle on N_g, and D be a connected orientable surface immersed in S^4 such that $\partial D = C$, and D is not tangent to N_g. Let ν_D be the normal bundle of D, then $\nu_D|_C$ is a solid torus with the unique trivialization induced from any trivialization of ν_D. Let $N_{N_g}(C)$ be the tubular neighborhood of C in N_g, then $N_{N_g}(C)$ is an twisted annulus or Möbius band in $\nu_D|_C$. We denote by $n(D)$ the number of right hand half-twist of $N_{N_g}(C)$ with respect to the trivialization of $\nu_D|_C$. Let $D \cdot F$ be mod-2 intersection number of D and F, $Self(C)$ be mod-2 double points number of C, and 2 be an injection $\mathbb{Z}_2 \to \mathbb{Z}_4$ defined by $2[n]_2 = [2n]_4$. Then the number $n(D)+2D\cdot F+2Self(C)$ (mod 4) depend only on the mod-2 homology class $[C]$ of C. Hence, we define

$$q_e([C]) := n(D) + 2D \cdot F + 2Self(C) \pmod 4.$$

This map q_e is called Guillou and Marin *quadratic form*, since q_e satisfies

$$q_e(x + y) = q_e(x) + q_e(y) + 2 < x, y >_2,$$

where $< x, y >_2$ means mod-2 intersection number between x and y. This quadratic form q_e is a non-orientable analogy of Rokhlin quadratic form.

A diffeomorphism ϕ over N_g is *e-extendable* if there is an orientation preserving diffeomorphism Φ of S^4 such that the following diagram is commutative,

$$\begin{array}{ccc} N_g & \xrightarrow{e} & S^4 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ N_g & \xrightarrow{e} & S^4. \end{array}$$

If the diffeomorphisms ϕ_1 over N_g is e-extendable, and ϕ_1 and ϕ_2 are isotopic, then ϕ_2 is e-extendable. Therefore, e-extendablility is a property about isotopy classes of diffeomorphisms over N_g. The group $\mathcal{M}(N_g)$ of isotopy classes of diffeomorphisms over N_g is called *the mapping class group of* N_g. An element ϕ of $\mathcal{M}(N_g)$ is *e-extendable* if there is an e-extendable representative of ϕ. By the definition of q_e, we can see that if $\phi \in \mathcal{M}(N_g)$ is e-extendable then ϕ preserves q_e, i.e. $q_e(\phi_*(x)) = q_e(x)$ for any $x \in H_1(N_g;\mathbb{Z}_2)$. What we would like to know is whether $\phi \in \mathcal{M}(N_g)$ is e-extendable when ϕ preserves q_e. But the answer for this problem would be depend on the embedding e. So, we will introduce an embedding which seems to be *simplest*.

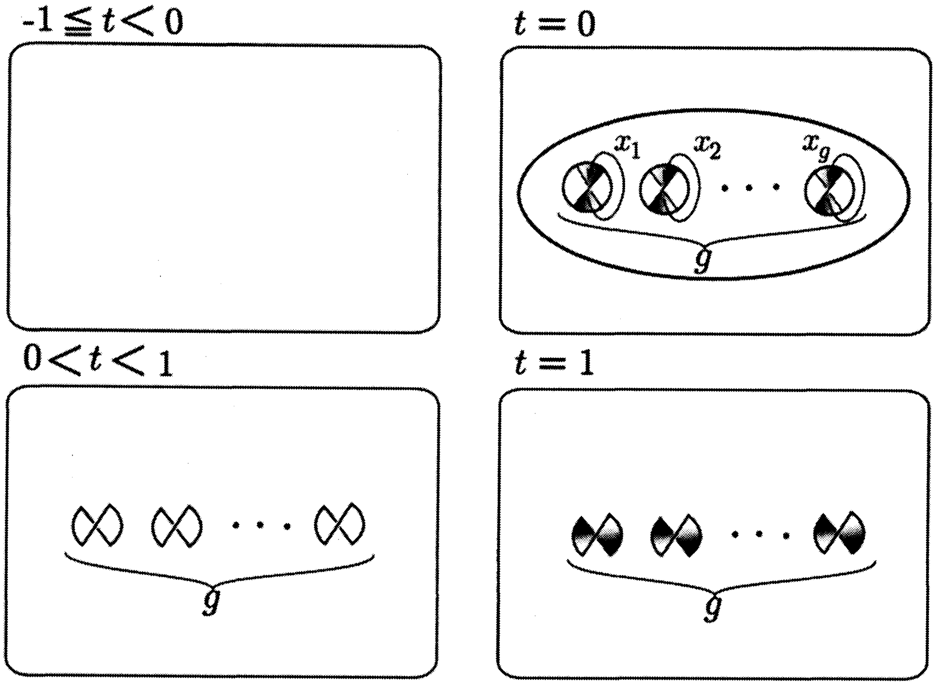


FIGURE 2

Let $S^3 \times [-1,1]$ be a closed tubular neighborhood of the equator S^3 in S^4. Then $S^4 - S^3 \times (-1,1)$ consists of two 4-balls. Let D^4_+ be the northern component of them, and D^4_- be the southern component of them. An embedding $ps : N_g \hookrightarrow S^4$ is *p-standard* if $ps(N_g) \subset S^3 \times [-1,1]$ and as shown in Figure 2. For the basis $\{e_1, \dots, e_g\}$ of $H_1(N_g; \mathbb{Z}_2)$ shown in Figure 2, $q_{ps}(e_i) = 1$. Since $< e_i, e_j >_2 = \delta_{ij}$, $q_{ps}(e_{i_1} + e_{i_2} + \cdots + e_{i_t}) = t$. The problem which we consider is the following:

Problem 2.1. If ϕ preserves q_{ps}, is $\phi \in \mathcal{M}(N_g)$ *ps*-extendable ?

In order to approach this problem, we review the generators for $\mathcal{M}(N_g)$.

3. Generators for $\mathcal{M}(N_g)$

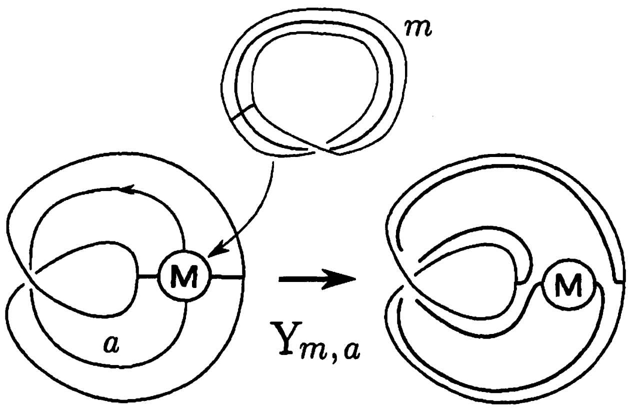


Figure 3. M with circle indicates a place where to attach a Möbius band

A simple closed curve c on N_g is A-circle (resp. M-circle), if the tubular neighborhood of c is an annulus (resp. a Möbius band). We denote by t_C the Dehn twist about an A-circle c on N_g. Lickorish [6] showed that $\mathcal{M}(N_g)$ is not generated by Dehn twists, and that Dehn twists and *Y-homeomorphisms* generate $\mathcal{M}(N_g)$. We review the definition of Y-homeomorphism. Let m be an M-circle and a be an oriented A-circle in N_g such that m and a transversely intersect in one point. Let $K \subset N_g$ be a regular neighborhood of $m \cup a$, which is homeomorphic to the Klein bottle with a hole, and let M be a regular neighborhood of m, which is a Möbius band. We denote by $Y_{m,a}$ a homeomorphism over N_g which may be described as the result of pushing M once along a keeping the boundary of K fixed (see Figure 3). We call

$Y_{m,a}$ a Y-homeomorphism. Since Y-homeomorphisms act on $H_1(N_g;\mathbb{Z}_2)$ trivially, Y-homeomorphisms do not generate $\mathcal{M}(N_g)$. Szepietowski [11] showed an interesting results on the proper subgroup of $\mathcal{M}(N_g)$ generated by all Y-homeomorphisms.

Theorem 3.1 ([11]). *$\Gamma_2(N_g) = \{\phi \in \mathcal{M}(N_g) \mid \phi_* : H_1(N_g;\mathbb{Z}_2) \to H_1(N_g;\mathbb{Z}_2) = id\}$ is generated by Y-homeomorphisms.*

In Appendix, we give a quick proof for this Theorem.

Chillingwirth showed that $\mathcal{M}(N_g)$ is finitely generated.

Theorem 3.2 ([1]). *$t_{a_1}, \ldots, t_{a_{g-1}}, t_{b_2}, \ldots, t_{b_{\lfloor \frac{g}{2} \rfloor}}, Y_{m_{g-1},a_{g-1}}$ generate $\mathcal{M}(N_g)$.*

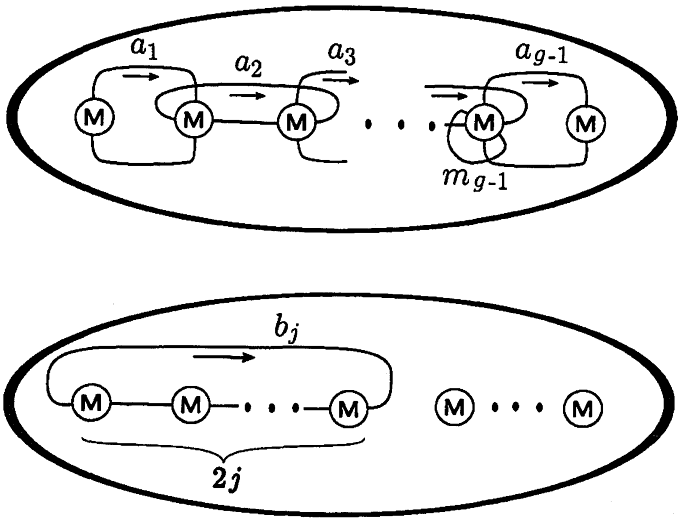


FIGURE 4

4. Lower genus cases

When genus g is at most 3, Problem 2.1 has a trivial answer.

The case where genus $g = 1$: $\mathcal{M}(N_1)$ is trivial.

The case where genus $g = 2$: $\mathcal{M}(N_2)$ is generated by two elements t_{a_1} and Y_{m_1,a_1}. Since the tubular neighborhood of a_1 in N_2 is a Hopf-band in $S^3 \times \{0\}$, t_{a_1} is *ps*-extendable by [4, §2]. Since a sliding of a Möbius band along the tube illustrated in Figure 5 is an extension of Y_{m_1,a_1}, Y_{m_1,a_1} is *ps*-extendable. Therefore, any element of $\mathcal{M}(N_2)$ is *ps*-extendable.

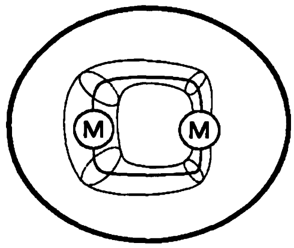


FIGURE 5

The case where genus $g = 3$: $\mathcal{M}(N_3)$ is generated by three elements t_{a_1}, t_{a_2} and Y_{m_2,a_2}. By the same argument as in the above case, it is shown that any element of $\mathcal{M}(N_3)$ is ps-extendable.

5. HIGHER GENUS CASES

In the case where genus $g = 4$, t_{b_4} does not preserve q_{ps} because $q_{ps}((t_{b_4})_*(x_1)) = q_{ps}(x_2 + x_3 + x_4) = 3 \neq 1 = q_{ps}(x_1)$. Therefore, t_{b_4} is not ps-extendable. We should consider the following subgroup of $\mathcal{M}(N_g)$,

$$\mathcal{N}_g = \{\phi \in \mathcal{M}(N_g) \mid q_{ps}(\phi_*(x)) = q_{ps}(x) \text{ for any } x \in H_1(N_g; \mathbb{Z}_2)\}\,.$$

In order to find a finite system of generators of $\mathcal{N}_g$, we introduce a group

$$\mathcal{O}_g = \{\phi_* \in Aut(H_1(N_g; \mathbb{Z}_2), <>_2) \mid \phi \in \mathcal{N}_g\}\,.$$

Then we have a natural short exact sequence

$$0 \to \Gamma_2(N_g) \to \mathcal{N}_g \to \mathcal{O}_g \to 0.$$

Since $\Gamma_2(N_g)$ is a finite index subgroup of $\mathcal{M}(N_g)$ and $\mathcal{O}_g$ is a finite group, theoretically, there is a finite system of generators for $\mathcal{N}_g$. But we would like to find an *explicit* system of generators. Nowik found a system of generators for $\mathcal{O}_g$ explicitly. For $a \in H_1(N_g; \mathbb{Z}_2)$, define $T_a : H_1(N_g; \mathbb{Z}_2) \to H_1(N_g; \mathbb{Z}_2)$ (transvection) by $T_a(x) = x + < x, a >_2 a$, where $<,>_2$ means mod-2 intersection form. We remark that if l is a simple closed curve on N_g such that $[l] = a \in H_1(N_g; \mathbb{Z}_2)$, then $(t_l)_* = T_a$. Nowik proved:

Theorem 5.1. *$\mathcal{O}_g$ is generated by T_a about a with $q_{ps}(a) = 2$.*

If we can find a finite system of generators for $\Gamma_2(N_g)$ explicitly, we can get a finite system of generators for $\mathcal{N}_g$. In the case where genus $g = 4$, we find that $\Gamma_2(N_4)$ is generated by the elements shown in Figure 6. Considering the action of Dehn twists corresponding to Nowik's generators of $\mathcal{O}_4$ on our system of generators for $\Gamma_2(N_4)$ by the conjugation, we see that $\mathcal{N}_4$ is generated by the 7 elements shown in Figure 7.

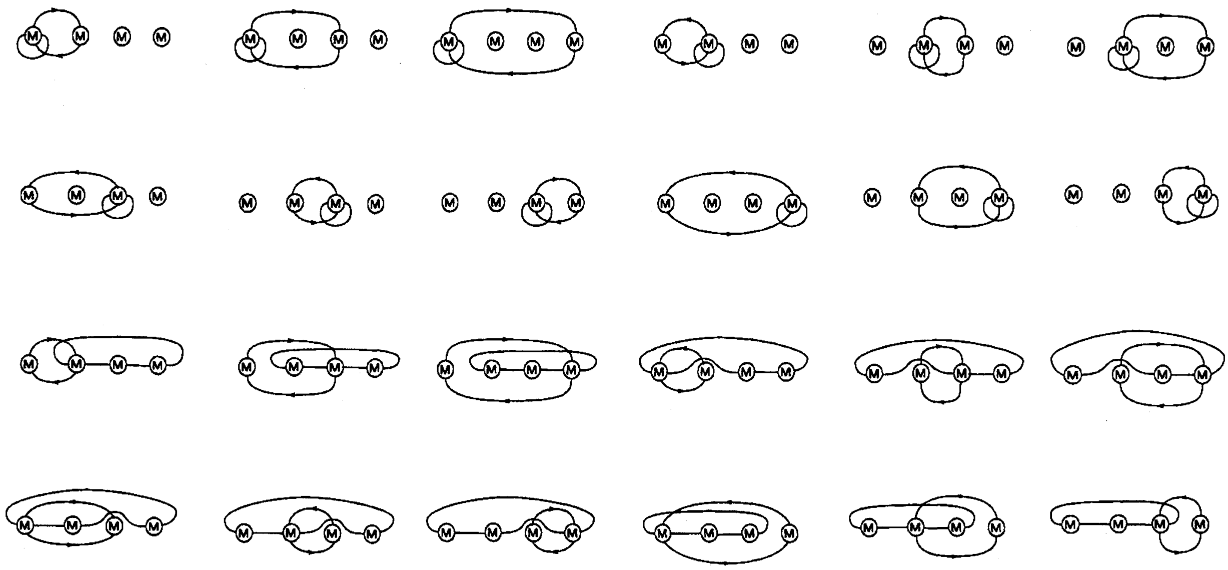


FIGURE 6

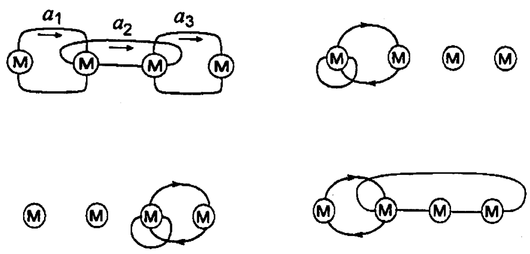
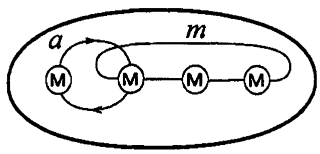


FIGURE 7

We can get an affirmative answer to Problem 2.1 when genus $g = 4$, if we answer the following Problem positively.



Problem 5.2. Is Y-homeomorphism $Y_{m,a}$ indicated in the above figure *ps*-extendable?

Appendix. A quick proof of Theorem 3.1

Nowik showed,

Theorem 5.3 ([10]). *$\Gamma_2(N_g)$ is generated by the following three types of elements:*
(1) $(t_c)^2$ about non-separating A-circles c (i.e., $N_g - c$ is connected) in N_g,
(2) t_c about separating A-circles c (i.e., $N_g - c$ is not connected) in N_g,
(3) Y-homeomorphisms.

This Theorem is proved by the same type of argument as in [5]. If we see that any elements of type (1) and (2) are products of Y-homeomorphisms, then we see Theorem 3.1. For (1), Szepietowski showed,

Lemma 5.4 (Lemma 3.1 of [11]). *For any non-separating A-circle c in N_g, $(t_c)^2$ is a product of two Y-homeomorphisms.*

Proof. There exists an M-circle m which intersects c in one point. Since $Y_{m,c}$ exchanges the two sides of c, we see $Y_{m,c}\, t_c\, Y_{m,c}^{-1} = t_c^{-1}$. Therefore, $(t_c)^2 = t_c(Y_{m,c}\, t_c\, Y_{m,c}^{-1})^{-1} = t_c\, Y_{m,c}\, t_c^{-1} \cdot Y_{m,c}^{-1} = Y_{t_c(m),c}\, Y_{m,c}^{-1}$. □

Let c be a separating A-circle, then at least one component F of $N_g - c$ is non-orientable. Let k be the genus of F.

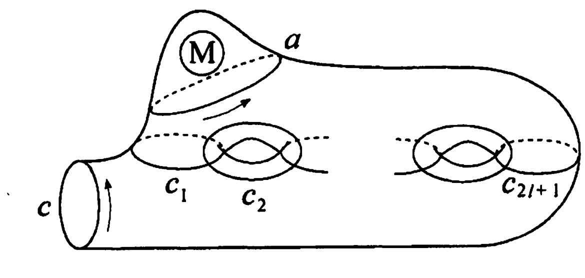


FIGURE 8

If k is an odd integer, we set $l = (k-1)/2$. Then F is as shown in Figure 8. By the chain relation, $t_a t_c = (t_{c_1} t_{c_2} \cdots t_{2l+1})^{2l+2}$. Let G_F be a subgroup of $\mathcal{M}(N_g)$ generated by $t_{c_1}, t_{c_2}, \ldots, t_{c_{2l+1}}$, and B_{2l+2} be the group of $2l+2$ string braid group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{2l+1}$ where σ_i is a braid exchanging the i-th string and the $i+1$-st string. Then there is a surjection $\pi : B_{2l+2} \to G_F$ defined by $\pi(\sigma_i) = t_{c_i}$. By this surjection, π(a full twist) $= (t_{c_1}t_{c_2}\cdots t_{2l+1})^{2l+2}$. Since a full twist is a pure braid and the subgroup of pure braids in B_{2l+2} is generated by $(\sigma_1\cdots\sigma_i)\sigma_{i+1}^2(\sigma_1\cdots\sigma_i)^{-1}$, $(t_{c_1}t_{c_2}\cdots t_{c_{2l+1}})^{2l+2}$ is a product of $(t_{c_1}\cdots t_{c_i})t_{c_{i+1}}^2(t_{c_1}\cdots t_{c_i})^{-1} = (t_{t_{c_1}\cdots t_{c_i}(c_{i+1})})^2$. By the above Lemma, $(t_{t_{c_1}\cdots t_{c_i}(c_{i+1})})^2$ is a product of Y-homeomorphisms. Since t_a is isotopic to the identity, t_c is a product of Y-homeomorphisms.

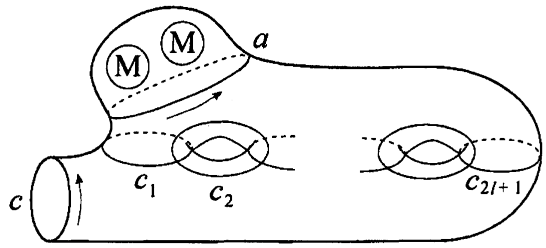


FIGURE 9

If k is an even integer, we set $l = (k-2)/2$. Then F is as shown in Figure 9. By the chain relation, $t_a t_c = (t_{c_1}t_{c_2}\cdots t_{2l+1})^{2l+2}$. By the same argument as above, we see that $t_a t_c$ is a product of Y-homeomorphisms. Let y be a Y-homeomorphism whose support is a Klein bottle with one boundary a, then $t_a = y^2$. Therefore, t_c is a product of Y-homeomorphisms.

Remark 5.5. Szepietowski showed that (2) is a product of Y-homeomorphisms in Lemma 3.2 of [11] by using the lantern relation.

REFERENCES

[1] **D. R. J. Chillingworth**, *A finite set of generators for the homeotopy group of a non-orientable surface*, Proc. Camb. Phil. Soc. 65 (1969), 409–430

[2] **L. Guillou and A. Marin**, *Une extension d'un théorème de Rohlin sur la signature*, C. R. Acad. Sc. Paris, t.285 (1977), Série A, 95–98

[3] **S. Hirose**, *Diffeomorphisms over surfaces trivially embedded in the 4-sphere*, Algebraic and Geometric Topology, 2, (2002), 791–824

[4] **S. Hirose and A. Yasuhara**, *Surfaces in 4-manifolds and their mapping class groups*, Topology, 47, (2008), 41–50

[5] **W.B.R. Lickorish**, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) 76 (1962), 531–540.

[6] **W.B.R. Lickorish**, *Homeomorphisms of non-orientable two-manifolds*, Proc. Cambridge Philos. Soc. 59(1963), 307–317.

[7] **W.B.R. Lickorish**, *On the homeomorphisms of a non-orientable surface*, Proc. Cambridge Philos. Soc. 61(1965), 61–64.

[8] **Y. Matsumoto**, *An elementary proof of Rochlin's signature theorem and its extension by Guillou and Marin*, in "À la Recherche de la Topologie Perdue", Progress in Math., 62(1986), 119–139

[9] **J.M. Montesinos**, *On twins in the four-sphere I*, Quart. J. Math. Oxford (2), 34(1983), 171–199

[10] **T. Nowik**, *Immersions of non-orientable surfaces*, Topology and its Applications 154(2007), 1881–1893.

[11] **B. Szepietowski**, *Crosscap slides and the level 2 mapping class group of a nonorientable surface*, preprint (arXiv:1006.5410)

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